More Work on Infinite Networks

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1 Introduction

In this paper I present a hodgepodge of ideas that I've had on infinite networks since completing my thesis. This document is in a very rought state, and certain aspects may be incorrect. I leave it the reader (and possibly myself in the future) to sort out these ideas and or expand on them to get some real results.

We attempt to generalize the notion of a supercritical halfplanar network to other networks which are nicely embedded in certain simply connected regions of the plane. At the current moment I have hope that this could allow us to recover certain infinite graphs embedded on surfaces by pulling the surface up to its simply connected covering space. We could potentially answer questions about extending functions on the finite embedded graph by extending cell sets of subsets of certain infinite medial graphs.

2 Medial Graphs

In a previous paper, we discuss and prove many results about a type of infinite network which we called supercritical halfplanar. We described these networks as a nice generalization of critical circular planar. Here we present a slightly broader generalization of critical circular planar.

Definition 2.1. Suppose that $\Omega \subseteq \mathbb{R}^2$ is a smoothly embedded two-dimensional submanifold of $\mathbb{R}^2$ with corners. If Ω is simply connected and has nonempty boundary, then we will call Ω a **superplane with boundary**.

We should note that by our definition a superplane with boundary is a closed subset $\Omega \subseteq \mathbb{R}^2$ such that $\overline{\operatorname{int}\Omega} = \Omega$ and $\partial\Omega$ consists of a union piecewise smooth arcs (where a piecewise smooth arc is the image of a function $\gamma : \mathbb{R} \to \mathbb{R}^2$ which is continuous on $\mathbb{R}$, differentiable at all but a discrete set of points, has nonvanishing derivative where it is defined, and has well defined, nonvanishing left and right hand derivatives wherever it fails to be differentiable).

Examples of superplanes with boundary are the closed unit disk, the upper half plane, an infinite closed strip in $\mathbb{R}^2$, and polygonal regions.

Definition 2.2. Let Ω be a superplane with boundary. Suppose that $\{g_i\}_{i\in I}$ is a collection of continuous functions with domains either $[0,1], [0,1)$ or $(0,1)$ or S^1 such that the following conditions are satisfied:

- The functions g_i are are all injective;
- The functions g_i are all proper maps (i.e. the inverse image of a compact set is compact)'
- The functions g_i are all smooth with nonvanishing derivative and have nonvanishing left and right derivatives at $t = 0$ and $t = 1$ if they are defined there;
- Each point in $\mathbb{R}^2$ is in the preimage of at most two functions g_i (i.e. at most two curves can intersect at a point);
- If g_i and g_j intersect, then they intersect transversally (i.e. their derivatives are linearly independent);
- If g_i is defined at 0 then $g_i(0) \in \partial\Omega$. Similarly if g_i is defined at 1 then $g_i(1) \in \partial\Omega$.;
- if $K \subseteq \mathbb{R}^2$ is compact, then only finitely many g_i have images which intersect K;
- if $t \in \operatorname{int}\operatorname{dom} g$ then $g(t) \in \operatorname{int}\Omega$.

If all of the above conditions are satisfied, then we say that $\{g_i\}$ is **geodesic system** and that the curves g_i are **geodesics**.

Definition 2.3. Suppose Ω is a superplane with boundary. we will define a **pseudo medial graph** on Ω to be the collection closures of connected components of $\Omega \setminus \bigcup_i g_i$ (where we naturally identify g_i with it's image). If all of the cells are compact and if the collection of cells is locally finite (any compact subset of $\mathbb{R}^2$ intersects only finitely many of them) then we say that M is a **medial graph**.

The closure of each connected component is called a **cell** and the cells which intersect $\partial\Omega$ are called **boundary cells**. The set of cells will be denoted by M and the set of boundary cells will be denoted by $\partial M \subseteq M$.

Definition 2.4. A geodesic system $\{g_i\}$ which no loops, lenses or self intersections is called **pseudocritical**. A pseudocritical geodesic system such that all geodesics have domain $[0,1]$ is called **supercritical**. A medial graph M is called pseudocritical (resp. supercritical) if it is induced by a pseudocritical (resp. supercritical) geodesic system.

Intuitively a supercritical medial graph is one where all of the geodesics have two endpoints on the boundary.

We define **corners**, **anticorners** and **degenerate corners** exactly as in the previous paper; a corner of a set X is where X contains a cell c adjacent to a medial vertex v, but none of the other three cells adjacent to v; an anticorner of X is a vertex v of the medial graph such that X contains three of the cells adjacent to v but not the fourth; a degenerate corner of X is when X contains exactly two of the cells adjacent to a vertex v, but these two cells are not adjacent.

If a set X has no anticorners we say it is **closed**. By a Zorn's lemma argument, if $X \subseteq M$ is a subset of a pseudocritical medial graph, we define the intersection of all closed sets containing X to be its closure, written $\overline{X}$. A Zorn's lemma argument shows that $\overline{X}$ can be written the union of a countable increasing chain of sets, each of which is obtained from the previous by filling in an anticorner. One observes that the intersection of closed sets is closed.

If g is a geodesic, then by the Jordan Curve Theorem g divides Ω into two disjoint regions. Similarly g divides M into two disjoint sets of cells, which we will call **half planes**. Half planes are obviously closed and hence intersections of half planes are closed. We will define a set $X \subseteq M$ to be **convex** if it is an intersection of halfplanes. If $X \subseteq M$ is any set of cells, we define $\widetilde{X}$ to be the intersection of all halfplanes that contain X.

Proposition 2.5. Let M be a medial graph induced by the geodesic system $\mathfrak{G} = \{g_i\}_{i \in I}$. If $X \subseteq M$ is a convex set, then X is itself a medial graph with superplane Ω_0 equal to the union of all the cells in X and geodesic system $\mathfrak{G}_0$ equal to a subset of appropriate restrictions of the g_i functions.

Lemma 2.6. Let M be a supercritical superplanar medial graph and let $c \in M$. If $g_1, \ldots, g_n$ are the geodesics neighborhing c and $H_1, \ldots, H_n$ are the half planes formed by the geodesics $g_1, \ldots, g_n$ respectively which contain c, then $\bigcap_{i=1}^n H_i = \{c\}$.

Proof. The intersection of half planes is connected (by induction), so it is sufficient to show that none of the cells neighboring c are contained in $\bigcap_{i=1}^n H_i$. But this is obvious by Jordan Curve Theorem nonsense since the geodesics adjacent to c divide c from a cell neighboring cell. □

Theorem 2.7. Let $X \subseteq M$ be a finite connected subset of a supercritical superplanar medial graph. Then X is contained in a finite submedial graph of M. In particular $\widetilde{X}$ is finite.

Proof. By definition, a finite submedial graph of M is just a finite convex subset of M. Hence it is sufficient to show that $\widetilde{X}$ is finite. Suppose that $c \in \widetilde{X}$, i.e. that if H is a half plane which contains X then H contains c. We first show that some geodesic g which neighbors c must also neighbor a cell in X. Suppose this were not the case, i.e. if all of the geodesics neighboring c did not also neighbor a cell in X. then if H_c^g denotes the half plane bounded by the geodesic g which contains the cell c, then for any geodesic g which neighbors c we would have that $X \subseteq H_c^g$ since by definition $c \in \widetilde{X}$ and $\widetilde{X}$ is the intersection of half planes containing X. But this would imply that $X \subseteq H_c^g$ for all geodesics g neighboring

c. By Lemma 2.6 we would have that if $g_1, \dots, g_n$ denote the geodesics bounding c, that

$$X \subseteq \bigcap_{i=1}^{n} H_c^{g_i} = \{c\}.$$

Since the theorem statement is trivial if X is empty, we can assume that $X = \{c\}$, which is trivially a finite submedial graph.

Hence if $c \in \widetilde{X}$, one of the geodesics neighboring c must also neighbor a cell in X. Since X is finite, there are only finitely many geodesics which neighbor cells in X. By supercriticality, each of these geodesics can only neighbor finitely many cells. A finite union of finite sets is finite, and hence the set of cells which neighbor a geodesic which neighbors a cell in X is finite. Since $\widetilde{X}$ is a subset of this set, it is in particular finite. □

Corollary 2.8. Let $X \subseteq M$ be a finite (not necessarily connected) subset of a supercritical superplanar medial graph. Then X is contained in a finite submedial graph of M.

Proof. For each pair of cells (c, c') in X there is a path of cell $P_{c,c'}$ of cells in X. Let Y be the union of the paths $P_{c,c'}$. Clearly $X \subseteq Y$ and Y is a finite connected set. Applying the previous lemma shows that $\widetilde{X} \subseteq \widetilde{Y}$ and $\widetilde{Y}$ is finite. □

We can actually modify the above argument to get a stronger result. First however, we need to define something. Obviously this parallels the development of the subject in Will Johnson's paper.

Definition 2.9. If c and c' are cells in a pseudocritical medial graph then define $d(c, c')$ to be the minimal length of any path between c and c'.

Remark 2.10. Note that we can alternatively define $d(c, c')$ to be the size of the smallest connected set of cells which contains both c and c'

We now prove a lemma that is verbatim a result from Will's paper:

Lemma 2.11. The quantity $d(c, c') - 1$ is equal to the number of geodesics which divide c from c'.

Proof. We note that by the local finiteness of our geodesics, a simple argument shows that only finitely many geodesics can divide c from c'.

Let P be a minimal length path. Let N denote the number of geodesics which separate c from c'. Since P can only cross one geodesic at a time, we know that $d(c, c') \geq N + 1$. To construct a path of length $N + 1$, simply proceed by induction on the number of geodesics which divide from c from c'. (The construction is identical to Will's paper). □

Corollary 2.12. In particular, any pseudocritical medial graph is connected (every pair of cells can be connected by a path of cells).

Theorem 2.13. Let $X \subseteq M$ be a finite connected subset of a pseudocritical medial graph. Then $\widetilde{X}$ is finite.

Proof. Let X be a finite connected subset of a pseudocritical medial graph M. Since X is finite, there can only be finitely many geodesics $g_1, \ldots, g_n$ which are adjacent to any cell in X. Suppose that c is a cell such that $d(c, c') \geq n + 2$ for any cell $c' \in X$. We claim that this implies that $c \notin \widetilde{X}$. Let c' be any cell in X. By the previous lemma, there are at least $n+1$ geodesics which divide c from c'. In particular, there is a geodesic which divides c from c' which is not adjacent to any cell in X. Since X is connected, and g is not adjacent to any cell in X, we know that g must divide c from X and hence $c \notin \widetilde{X}$. Since there are only finitely many cells in M of distance at most $n+1$ from X, we know that $\widetilde{X}$ is finite. □

Theorem 2.14. If M is a pseudocritical medial graph and $X \subseteq M$ is a finite subset which doesn't intersect the boundary, then X has at least three corners.

Proof. By induction as in Will's paper. □

Theorem 2.15 (The Filling Lemma)**.** The filling lemma applies to pseudocritical superplanar graphs.

Proof. Simply copy and paste from my previous paper. □

Corollary 2.16. If $X \subseteq M$ is closed and connected then $\widetilde{X} = X$.

Proof. Copy paste. □

Corollary 2.17. If $X \subseteq M$ is connected then $\overline{X} = \widetilde{X}$.

Proof. Even more copy paste! □

3 Recovery

In general there is no known way to recover a general infinite pseudocritical graph, even if it is embedded in the upper half plane. In a previous paper we show how to recover supercritical half planar networks, and here we generalize this slightly to supercritical superplane networks with a certain property:

Definition 3.1. We will say a superplane Ω has finite boundary if $\partial\Omega$ has a finite number of components.

Definition 3.2. We will say that an electrical network is supercritical finite superplanar if it is supercritically embedded in a finite boundary superplane.

Lemma 3.3. Suppose M is a supercritical finite superplanar medial graph embedded in Ω and suppose that Ω is unbounded. Let F be one of the components of the boundary. As discussed above F is the image of a continuous piecewise smooth curve $\gamma : (-\infty, \infty) \to \mathbb{R}^2$. There exist t_0 and t_1 such that $t_0 < t_1$ and two boundary components F_0 and F_1 such that if g is a geodesic with an endpoint in the image of $(-\infty, t_0)$ then the other endpoint of g is in F or F_0, and similarly if g is a geodesic with an endpoint in the image of (t_1, ∞) then the other endpoint is on F or F_1.

Proof. This is a proof by induction on the number of boundary components. The $n = 2$ case is obvious. If all geodesics on F return to F, then we are done. Let g be an arbitrary geodesic with an endpoint on F and an endpoint on another boundary component F'. Consider the two half planes H and H' generated by g. Each of the other $n - 2$ components of the boundary must completely be contained in one of the half planes H and H'. Viewing H and H' as submedial graphs, we observe that each H and H' have no more than $n - 1$ boundary components. Let s_0 denote the real number such that $\gamma(s_0) = g(0)$. Without loss of generality, suppose that H is the half plane which contains $\gamma(s)$ for $s > s_0$. Now the reader verifies that the boundary component of H which contains this portion of F is parametrized piecewise by a restriction of γ, the geodesic g, and a restriction of another parametrization for the other component of the boundary which g intersects. Let γ' bet the parametrization of this new boundary component. By considering either large s or small s we can restriction to the image of γ' which is also in the image of γ. Without loss of generality suppose that we can do this by considering small s. Since only finitely many geodesics can cross g, by considering s small enough that $\gamma'(s)$ never intersects a geodesic which crosses g, we can now apply the inductive hypothesis to H, with the boundary component in consideration being the image of γ'. □

4 Cutpoint Lemma for Pseudocritical Networks.

Here we prove a version of the cutpoint lemma for infinite pseudocritical graphs. We refer the reader to my thesis for the definition of a pseudocritical graph. We include the additional assumption (which should be ever present) that all medial graphs have the property that all cells are compact and that all edges of the medial graph are also compact. The second assumption actually follows from the first, assuming we have some other normality assumptions, but we won't concern ourselves with such pedantic nonesense.

As a historical note, one should recall that the Cutpoint lemma was first proven for finite critical circular planar networks using a purely geometric argument. The argument appears in [1]. In my thesis, I gave an alternate proof using special γ-harmonic functions on the network. This adapted to an argument for the infinite case, yielding the cutpoint equality for supercritical graphs and the cutpoint inequality for pseudocritical graphs. Here we present the cutpoint equality for pseudocritical graphs using a geometric argument.

Let M be an infinite pseudocritical medial graph and let (x, y, R) be a directed segment (using the notation from my thesis). This just means that $x < y$ and the points x and y lie on the real axis and R is the closed interval $[x, y]$. Let $b(R)$ denote the number of black cells completely contained in R. Let $r(R)$ denote the number of geodesics which have both endpoints in R. Finally let $m(R)$ denote the largest k such that there is a k-connection which respects the cut R (or using the notation from my thesis, a k-flowout which respects R).

We will prove the cutpoint lemma for certain pseudocritical networks, namely the connected ones with compact cells in the medial graph. We will first need several lemmas.

Lemma 4.1. Let M be a pseudocritical medial graph and suppose the partitioned graph G is medially embedded in M. Let c be a compact dual cell in M. If ∂c has either one or zero boundary arcs corresponding to closed intervals of $\mathbb{R}$, then all primal cell adjacent to a particular dual cell can be connected by a path in G. If ∂c has more than one boundary arc corresponding to a closed interval of $\mathbb{R}$, then G is not connected.

Proof. Left to reader. See Figure 1.

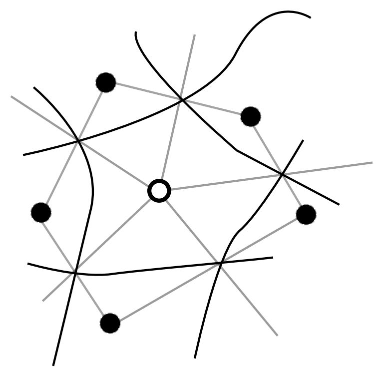

Figure 1: An example of how the primal cells are connected to each other in Lemma 4.1.

□

Corollary 4.2. Suppose G is a partitioned graph that is medially embedded in the pseudocritical medial graph M. If G is connected and c is a compact cell dual cell of M, then all primal vertices medially adjacent to c can be connected to each other through a path of cells which are adjacent to c.

Lemma 4.3. Let M be a pseudocritical medial graph with compact cells and the paritioned graph $G = (\partial G, \operatorname{int} G)$ is medially embedded in M. Suppose further that G is connected. Let $F \subseteq M$ be any finite collection of cells which contains a boundary cell. Then there is a path in G which avoids all primal cells of F and which starts at some boundary vertex which is to the left of all boundary cells in F and ends at a boundary vertex in G.

Proof. Let L be the set of vertices of G which can be connected to a primal vertex of G (through a path through G) which is to the left of all boundary vertices of F. Let R be the set of vertices of G which can be connected to a primal vertex of G which is to the right of all boundary vertices of G. We will show that $R \cap L$ must be nonempty. Let $\mathfrak{L}$ denote the set of all primal cells corresponding to vertices of L, taken along with the set of all dual cells which are adjacent to a cell in L and are not in F and not adjacent to any cells in F. Define $\mathfrak{R}$ similarly.

We will be done if we can show that $R \cap L$ is nonempty, so suppose to the contrary that $R \cap L = \varnothing$. We first claim that this implies $\mathfrak{R} \cap \mathfrak{L}$ is empty. Suppose that there is a cell c in $\mathfrak{R} \cap \mathfrak{L}$. Clearly c cannot be a primal cell, and hence c must be a dual cell. But then c is a dual cell which neighbors a cell in R and a cell in L. Since c does not neighbor any cell in F, and by Corollary 4.2, we know that all primal cells neighboring c can be connected by a path which is not in F.

Let F' denote the set of all cells which are in F or adjacent to a cell in F. Let $\mathfrak{F}$ denote the union of all cells in F' (as a subset of $\mathbb{R}^2$). Since each cell in F' is compact and F' is finite, we know that $\mathfrak{F}$ is compact. Since $\mathfrak{F}$ is compact, there is a path $\gamma : [0, 1] \to \mathbb{H} \setminus \mathfrak{F}$ such that $\gamma(0)$ is a boundary vertex of G which lies in L and $\gamma(1)$ is a boundary vertex of G which lies in R. Furthermore, since the set of medial vertices is locally finite, we can assume that γ avoids all medial vertices (we remind the reader that a medial vertex is the intersection point of two geodesics). Also, by some differential geometry nonesense, we can assume that γ is transverse to the set of geodesics with the geodesic vertices removed (which is a smooth 1 dimensional manifold). This implies that the inverse image of any geodesic is a discrete subset, and hence a finite number of points. Hence a simple argument shows that this implies that if γ crosses a geodesic at $s_0 \in (0, 1)$ then $\gamma(t)$ is in the interior of a medial cell in an interval $(s_0 - \epsilon, s_0)$ and in the interior of a *different* cell in an interval $(s_0, s_0 + \epsilon')$ for some ϵ and ϵ'. Define the set $S = \bigcup_{c \in \mathfrak{L}} c$. Since S is a locally finite union of closed sets, we know that S is closed. By continuity, the set

$$\gamma^{-1}(S) = \{t \in [0, 1] : \gamma(t) \text{ is in a cell in } \mathfrak{L}\}$$

is closed. Let $t_0 = \sup \gamma^{-1}(S)$ Since the above set is closed (since $S = \bigcup_{c \in \mathfrak{L}} c$ is the union of cells in $\mathfrak{L}$ is a locally finite union of closed sets and hence is closed, and the above set is just $\gamma^{-1}(S)$. Since $\gamma^{-1}(S)$ is closed and bounded, it is compact, and hence there is a maximum element, t_0, of $\gamma^{-1}(S)$. Furthermore, clearly we must have $t_0 < 1$, since otherwise $\gamma(1)$ would be a primal vertex which is in both R and L. Now our transversality assumption implies that $\gamma(t)$ is in the interior of a cell c for t slightly less than t_0 and that $\gamma(t)$ is in a *different* cell, c' for t slightly larger than t_0. Clearly c and c' must be adjacent cells. Clearly $c \in L$ and $c' \notin L$. If c is a dual cell, then clearly c' is a primal cell. If $c \in L$, then c is adjacent to a primal cell in L by definition, and hence by Corollary 4.2 there is a path from a cell in L to c' which does not pass through any cells in F, and hence $c' \in L$. But this is a contradiction. Similarly if c is a primal cell, then c' is a dual cell which, by construction of F', is not adjacent to any cell in F, and hence $c' \in L$. But these are the only possible cases, and hence we must conclude that the maximum value of γ is 1, which implies that there is a primal vertex in L and R, which implies that there is a path of primal cells through G from a boundary vertex to the left of all boundary vertices of in F to a boundary vertex which is to the right of all boundary vertices of F. This is exactly what we wanted to construct, so we are done.

□

Lemma 4.4. Suppose M is a pseudocritical medial graph with compact cells and $F \subseteq M$ is a finite subset of cells and F contains a boundary cell in the medial graph. Then there is a path of medial cells in $M \setminus F$ from a boundary cell which is to the left of all boundary cells in F to a boundary medial cell to the right of all boundary cells in F.

Proof. Suppose F is a finite set of cells and let $\mathfrak{F}$ denote the union of points in $\mathbb{R}^2$ which are in cells in F. By assumption $\mathfrak{F}$ is compact. Construct a path $\gamma : [0, 1] \to \mathbb{H}$ such that $\gamma(0)$ is to the left of any boundary cell in F and $\gamma(1)$ is to the right of any boundary cell in F and γ doesn't enter any cell in F. By basic theorems about transversality of manifolds, we can assume that γ doesn't cross any medial vertex (a point where two geodesics cross) and that γ is transverse to the set of all geodesics. This last assumption implies that γ properly crosses any geodesic that it intersects (i.e. if $\gamma(t_0)$ is a point on a geodesic, then $\gamma(t)$ is in the interior of a cell for t slightly less than t_0 and $\gamma(t)$ is in the interior of an adjacent (but different) cell for t slightly larger than t_0). The curve γ determines a path of geodesic cells. The details are left to the reader, though the reader should look at [5] for a background in transversality, or just ignore the details since they are essentially irrelevant. □

We now prove the cutpoint lemma for connected pseudocritical networks with compact cells. The reader should familiarize themself with the argument in [6] for finite networks since we are essentially just adapting that proof to the infinite case. We construct a function ϕ_g for each reentrent geodesic and we construct a map L which takes boundary voltages inside of our cut R to boundary currents outside of R. We apply the rank nullity theorem of lineary algebra to L and show that the functions ϕ_g span the kernel of L. In the supercritical case, these functions ϕ_g span the kernel because of simple geometric considerations (we just look at the geodesic closure of where a function in the kernel is zero). Unfortunately we have to work a little harder in the pseudocritical case, but everything works out in the end.

Theorem 4.5. Let M be a pseudocritical halfplanar medial graph with compact cells and let $\Gamma = (G, \gamma)$ be a connected electrical network embedded in M. Let (x, y, R) be a directed segment. Then the Cutpoint Lemma is satisfied:

$$b(r) = r(R) + m(R).$$

Proof. Let X_ℓ denote the set of boundary cells to the left of R and let X_r denote the set of boundary cells to the right of R. Let $\widetilde{\phi}$ denote a voltage and covoltage function for ϕ defined on all of the cells of the medial graph, such that the covoltage on X_ℓ is zero. By assummption we know that the covoltage on X_r is constant (though *a-apriori* not necessarily zero).

We will show first that ϕ is zero except at finitely many vertices. To do this, we will have to use Lemma 4.3 repeatedly. Let F_0 be the set of primal boundary vertices at which ϕ is nonzero or has nonzero current leaving. We now will proceed as in the [6]. Let $v_1, \dots, v_n$ denote the boundary primal vertices which are completely contained in the cut R. Let V be the n-dimensional vector space of minimal functions ϕ such that $\phi|_{\partial G}$ is zero everywhere except possibly on the set $\{v_1, \dots, v_n\}$. Let ∂G° denote the set of boundary vertices which are completely contained in $\mathbb{R} \setminus R$. Let V_C denote the vector subspace of $\mathbb{R}^{\partial G^\circ}$ consisting of ∂G°-tuples of currents leaving vertices in ∂G° from functions in V. Note that V_C is finite dimensional and has dimension at most n. Now define the map $L : V \to V_C$ sending a minimal function $\phi \in V$ to the ∂G°-tuple of currents leaving the vertices in ∂G°. Since V and V_C are finite dimensional vector spaces, we have that

$$\dim V_C = \dim \ker L + \operatorname{rank} L.$$

Notice that $\operatorname{rank} L$ equal to the maximum k-flowout which respects the directed segment R. Furthermore, $\dim V_C$ is just the number of black vertices completely contained inside the cut R. Thus we have that

$$b(R) = \dim \ker L + m(R).$$

We now need to show that $\dim \ker L = r(R)$. As shown in [6], each reentrant geodesic determines a minimal function, and the set of these minimal functions is linearly independent. Each of these functions is supported only inside of the region bounded by a reentrant geodesic, and also as shown in [6], any minimal voltage-covoltage function which is supported only on $\bigcup_{g \text{ recurrent}} B(g)$ is a unique linear combination of the functions above. Hence we have that $\dim \ker L \geq r(R)$. Let $g_1, \dots, g_m$ be the reentrant geodesics of the directed segment R (hence $m = r(R)$).

Now suppose that $\widetilde{\phi}$ is a voltage-covoltage function in $\ker L$ which has finite support on the medial graph. Let F be the set of cells where $\widetilde{\phi}$ is nonzero. By Lemma 4.4,we know that there is a path P of cells in the medial graph from a boundary cell that is to the left of all boundary cells in F, to a cell that is to the right of any boundary cell in F. Define $X = X_r \cup X_\ell \cup P$. Since X is connected, we know by Corollary 2.17 that $\overline{X} = \widetilde{X}$. Since each reentrant geodesic divides the half plane into two regions, exactly one of which is bounded, if g_i is a reentrant geodesic, we will let $B(g_i)$ denote the set of cells in the bounded region determined by g_i, and we will let $U(g_i)$ denote the unbounded set of cells determined by g_i. It is easy to verify that $\widetilde{X}$ is equal to exactly $\bigcap_{i=1}^m U(g_i)$. By basic facts about γ-harmonic functions, we thus know that $\widetilde{\phi}$ is supported at most on the set

$$\left(\bigcap_{i=1}^m U(g_i) \right)^c = \bigcup_{i=1}^m B(g_i).$$

By what we have shown, this would imply that ϕ is a unique linear combination of the functions determined by our geodesics.

Hence to show that $\ker L = r(R)$, it is sufficient to show that all functions in $\ker L$ have a voltage-covoltage extension which is finitely supported.

We will now do this. Let $\phi \in \ker L$ be a minimal function in the kernel of L. By Lemma 4.3 there is a path from a boundary vertex v to the left of R to a boundary vertex v' to the right of R which and which avoids all of the boundary vertices in R. A simple argument shows that we can assume that this path passes only through the interior of the graph. By deleting cycles we can assume that this path has no repeated vertices. This path determines exactly two regions by the Jordan curve theorem, exactly one of which is bounded. The bounded region will contain all boundary vertices with nonzero voltage. We will consider the voltages on this path. Let S denote all of the vertices in the bounded region (including those on the path P). If all voltages on this path are zero, then we consider the function $\psi : V \to \mathbb{R}$ defined by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \in S \\ 0 & \text{if} v \notin S \end{cases}.$$

A simple argument shows that $\psi|_{\partial G} = \phi|_{\partial G}$ and $P(\psi) \leq P(\phi)$ and that $P(\psi) < P(\phi)$ if any vertex in S^c has nonzero voltage. Since ϕ is minimal power, we know that $\phi = \psi$. Hence if all voltages along P are zero, then we will be done, since ϕ will have finite support. Hence some vertex along P must have nonzero voltage. We now perform the trick that we did in proving that certain maps have nonzero determinant in [6]. Let v_0 be the last vertex in P which has voltage zero. Connected to this vertex must be a vertex of positive voltage and a vertex of negative voltage by the local maximum principle. Construct paths α_1 and β_1 as follows. Let α_1 and β_1 start along the path P until the vertex v_0. After v_0, construct α and β inductively by picking the next vertex in α to be the neighbor to the current vertex with the minimum voltage, and picking β_1 to be the maximum. Hence α_1 is a path of vertices with nonincreasing voltages, which is strictly decreasing after v_0. Similarly β_1 is a path of vertices with nondecreasing voltgages, which is strictly increasing after v_0. Clearly α_1 and β_1 do not intersect after they diverge at v_0, and furthermore a simple argument shows that there are no loops in α_1 or in β_1 since the voltages are eventually strictly decreasing or increasing.

We now claim that both α_1 and β_1 must terminate at a vertex on the boundary. We will first show that at least one of the paths terminates on the boundary. Suppose that neither do. Notice that the paths α_1 and β_1 determine three regions of A_1, A_2 and A_3, two of which contain boundary cells. Label the regions as in the below Figure 2 (and possibly multiply ϕ by -1 to switch α and β)

Now if we have this configuration, we just do the standard trick and define $\psi : V \to \mathbb{R}$ by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \in A_3 \\ \phi(v) & \text{if } v \notin A_3 \text{ and } \phi(v) \leq 0 \\ 0 & \text{if } v \in\in A_3 \text{ and } \phi(v) > 0 \end{cases}.$$

A simple argument shows that $\phi|_{\partial G} = \psi|_{\partial G}$ and $P(\psi) \leq P(\phi)$. Furthermore,

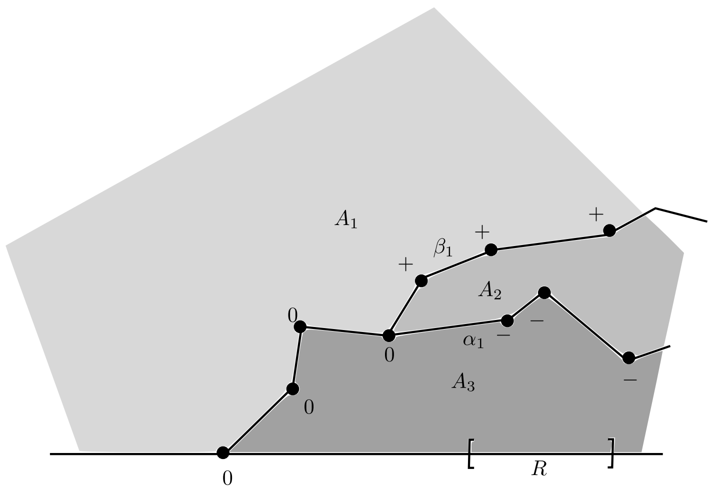

Figure 2: The curves α_1 and β_1, assuming neither terminate, and the regions A_1, A_2 and A_3.

the reader checks that in fact $P(\psi) < P(\phi)$, contradicting the minimality of ϕ. Hence at least one of α_1 and β_1 must terminate.

We now will show that both α_1 and β_1 must terminate. Suppose that exactly one terminates. By possibly multiplying by -1 we will assume that α_1 terminates but β_1 does not. Since our graph embedding is locally finite, we know that if we parametrized the edges and vertices of β_1 as a topological path, then that path would go to ∞. Hence we have the situation as in Figure 3.

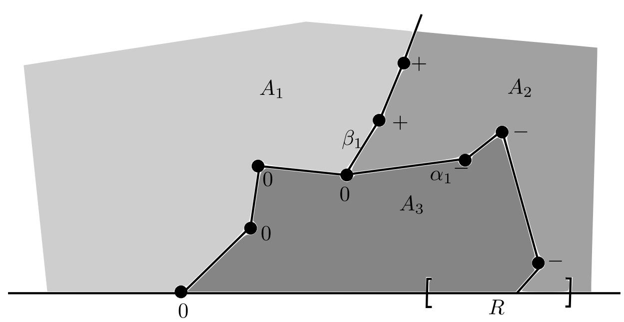

Figure 3: The curves α_1 and β_1, assuming that α_1 termiantes but β_1 does not.

We note that by the maximum principle, every vertex in the region A_1 must have nonnegative voltage (define a function ψ by changing negative values to zero in the region A_1 and observe that it reduces power iff ϕ has a vertex of properly negative voltage in A_1). Now let F denote the set of all vertices in the path α_1 or in the region A_3 or in the cut R. Since F is a finite set of vertices, Lemma 4.3 implies that there is a path P' from a boundary vertex to the left of all boundary vertices if F to a boundary vertex which is to the right of all boundary vertices of F. An argument involving planarity shows that this path must cross a vertex of β_1 which is not shared with α_1. This implies that the voltage along the path P' is not constantly zero. Consider the last vertex, v' where the voltage is zero. It is possible that v' is on the boundary, but by assumption if v' is on the boundary, then there is no current flowing leaving v' (since ϕ is in the kernel of L). If v' is an interior vertex, then we also would have that the net current leaving v' is zero. In all cases, v' would have net current 0 leaving it. Since the voltages in A_1 are nonnegative, we know the next vertex in the path P' must have positive voltage. By the local maximum principle there must also be a vertex of negative voltage adjacent to v'. But clearly all vertex adjacent to v' must have positive voltage (since adjacent vertices must be in A_1 or be along β_1). This is a contradiction and hence β_1 must terminate.

We now observe that both α_1 and β_1 must terminate inside R, since the only boundary vertices with nonzero voltages are inside R. Now apply Lemma 4.3 to get a path P_2 which avoids α_1, β_1 and R which starts to the left of R and ends to the right of R. Again we can assume that P_2 has no repeated vertices. Now if all vertices have zero voltage along P_2 then we are done, so assume otherwise. Construct α_2 and β_2 as above. An identical argument shows that α_2 and β_2 must both terminate on the boundary. Now construct a path P_3 which avoids $\alpha_1, \beta_1, \alpha_2, \beta_2$ and R and which starts to the left of $R, \alpha_1, \beta_1, \alpha_2$ and β_2 and ends to the right of $R, \alpha_1, \beta_1, \alpha_2$ and β_2. Construct α_3 and β_3 as above. They must terminate by the same argument. Keep constructing paths α_i and β_i. Note that α_i can never intersect β_j if $i \neq j$. Let $r(\alpha_i)$ denote the last vertex of α_i (as a real number), and define $r(\beta_i)$ similarly. Note that

$$r(\alpha_1) < r(\beta_2) < r(\alpha_3) < \cdots .$$

Since $r(\alpha_i)$ and $r(\beta_i)$ are all vertices of R, and $r(\alpha_1) < r(\beta_2) < r(\alpha_3) < \cdots$, we know that eventually in constructing all of these paths α_i and β_i, we must eventually exhaust all of the vertices in R. Whenever we construct a path P_i as above which has a vertex with nonzero voltage, we will be able to construct paths α_i and β_i, which must terminate. Since there cannot be infinitely many paths of the form α_i or β_i since there are only finitely many vertices in R, we know that eventually we will construct a path P_j of the above form which has voltage zero along it. But, as we've already remarked, the existence a path P_j of the above form which has voltage zero along it implies that ϕ has finite support.

By what we've said already, we know that this implies that ϕ is in the span of the set of functions which are supported inside of a single reentrant geodesic. The reader now verifies that this implies the statement of the cutpoint lemma. □

4.1 Facts About Functions with Finite Support

We remark that the reader should pay close attention to the proof of the Pseudocritical Cutpoint Lemma (Theorem 4.5). There are several interesting corollaries that we get immediately from the proof:

Corollary 4.6. Suppose M is a pseudocritical half planar medial graph with compact cells and $\Gamma = (G, \gamma)$ is an embedded electrical network. Then any minimal function ϕ with finite support on the boundary of the network (both voltage and currents equal to zero except at finitely many boundary vertices) has support contained inside of the union of the bounded regions bounded by reentrent geodesics. In particular, a network Γ as above has minimal functions with finite support iff there exist reentrent geodesics.

The last statement of the above corollary was conjectured by Gracie Ingermanson.

5 Well Connectedness

Here we begin a discussion of an property called well connectedness. The main theorem, that a pseudocritical half planar graph with no reentrent geodesics is well connected in a certain sense, was conjectured by Gracie Ingermanson. Her work on the theorem proceeded by an uncrossing argument using the medial graph. We will give a proof based on linear algebra using the techniques we used to prove the cutpoint lemma.

Definition 5.1. Let Γ be a half planar electrical network. We will say that Γ is circularly well connected if for all k-pairs $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ of primal boundary vertices such that $a_1 < a_2 < \cdots a_k < b_1 < \cdots < b_k$ there is a k connection from A to B. We will call such a k connection a **circular k-connection**.

Definition 5.2. We will say that a half planar electrical network Γ is **conformally circularly well connected** if it is circularly well connected and if for all k-pairs $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ of primal boundary vertices such that $a_1 < \cdots a_j < b_1 < \cdots b_k < a_{j+1} < \cdots a_k$ for some $1 \leq j \leq k$ there is a k-connection from A to B. We will call such a k-connection a **conformally circular k-connection.**

Definition 5.3. Suppose M is a pseudocritical half planar medial graph with compact cells and suppose that $\Gamma = (G, \gamma)$ is medially embedded in M and that M has no reentrent geodesics. Then we will say that Γ is an **ultralattice**.

Lemma 5.4. Let Γ be an ultralattice and let $A = (a_1, \ldots, a_k)$ be a collection of k primal boundary vertices such that $a_1 < \cdots < a_k$. Let c be an arbitrary primal boundary vertex such that $c < a_1$ and let $\mathfrak{L}$ be the collection of all primal vertices to the left of c (and including c). Then there is a k-connection from A to $\mathfrak{L}$.

Proof. We proceed in the only way imaginable. Define the map $L : \mathbb{R}^A \to \mathbb{R}^{\mathfrak{L}}$ which sends an A-tuple v of real numbers to the $\mathfrak{L}$-tuple of currents leaving the vertices of $\mathfrak{L}$ from the minimal function which has boundary values v on A and is zero everywhere else on the boundary. By a theorem in [6], it is sufficient to show that L is full rank. Let V denote $\mathbb{R}^A$ and let C denote the image of L in $\mathbb{R}^{\mathfrak{L}}$. With a slight abuse of notation, we will now write L as a map from V to C. To show that L is full rank, we just need to compute the kernel of L. Specifically we will show that the kernel of L is just 0. Let ϕ be in the kernel of L. By Corollary 4.6, if we can show that ϕ has finite support, then we will know that $\phi = 0$ since M has no reentrent geodesics.

Hence we will just show that ϕ has finite support. Let F_0 be the set of all primal boundary vertices between c and a_k. By Lemma 4.3, there is a path P_1 in G from some boundary vertex in $\mathfrak{L}$ to some primal boundary cell to the right of F_0 which passes only through the interior of G. Now let F_1 denote the union of F_0 and all vertices in this path. Apply Lemma 4.3 to get another path P_2 from a boundary vertex in $\mathfrak{L}$ to the left of all primal vertices in F_1 to some primal boundary vertex to the right of F_1. Continue in this manner and construct k disjoint paths $P_1, \ldots, P_k$, each of which starts at a primal boundary vertex in $\mathfrak{L}$ and ends at a primal boundary vertex to the right of A, and each of which only passes through the interior of G. Without loss of generality, we may assume that the paths P_i have no repeated vertices, and hence no loops. The situation is shown in Figure 4.

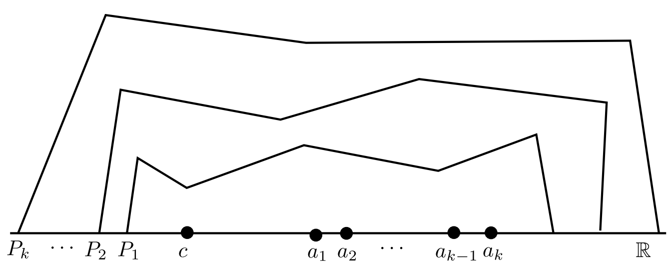

Figure 4: The paths $P_1, \ldots, P_k$ with respect to the vertices $a_1, \ldots a_k$ and c.

From each primal vertex, pick an adjacent vertex with maximal voltage (in the case where two vertices both attain the maximal voltage, pick arbitrarily). Similarly, for every primal vertex, pick an adjacent vertex with minimal voltage amongst neighboring cells and resolve ambiguity arbitrarily. Now ϕ is zero completely on any of the paths P_i, then we will be done, since if we let U denote the primal vertices outside of the bounded region bounded by P_i, then if ϕ is zero on P_i, we note that the function $\psi : V \to \mathbb{R}$ defined by

$$\psi(v) = \begin{cases} \phi(v) & \text{if } v \in U^c \\ 0 & \text{if } v \in U \end{cases}$$

satisfies $P(\psi) \leq P(\phi)$ and $\phi|_{\partial G} = \psi|_{\partial G}$, implying that $\phi = \psi$ since ϕ is minimal.

Hence ϕ would have finite support, and by what we've shown, we would be done. Hence we know that there each P_i must have a vertex which has nonzero voltage. Let v_i be the last vertex of P_i such that ϕ is zero all the way from the first vertex in P_i to v_i. Since the sum of the currents of ϕ at v_i must be zero (either v_i is the first vertex in P_i and hence there is zero net current leaving v_i by assumption or v_i is an interior vertex), we know by the local maximum principle that there is a vertex adjacent to v_i of positive voltage, and a vertex adjacent to v_i of negative voltage. Construct the paths α_i and β_i by following the least adjacent voltage and the greatest adjacent voltage respectively (as in the proof of the Cutpoint Lemma for the pseudocritical cutpoint lemma). An identical argument to the one from the cutpoint lemma shows that all the paths α_i and β_i must terminate on the boundary. Let $F(\alpha_i)$ denote the final vertex of the path α_i and let $F(\beta_j)$ denote the final vertex of β_j. Note that the vertices $F(\alpha_j)$ and $F(\beta_j)$ must be amongst $\{a_1, \ldots, a_k\}$ since those are the only boundary vertices with nonzero voltage. The reader checks that since α_i and β_j cannot cross if $i \neq j$, and since α_i and β_i must have distinct endpoints, there is no way for all the α_i and β_j to terminate, and hence we have a contradiction. The details are as in [6], so we omit them. □

Theorem 5.5. Suppose that Γ is an ultralattice with medial graph M. Then Γ is circularly well connected.

Proof. Let $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_k)$ be two collections of boundary vertices such that $a_1 < \cdots < a_k < b_1 < \cdots b_k$. We do the standard nonsense. Define the map L from $\mathbb{R}^A$ to $\mathbb{R}^B$. Define L by sending an A-tuple v to the B-tuple of currents leaving the vertices in B from the minimal function with boundary values v on A and 0 on the rest of the boundary. By a theorem in [6], if L has full rank, then there will be a k-connection from A to B. It is sufficient to show that $\ker L = 0$. Suppose that $\phi \in \ker L$. Suppose without loss of generality that A is to the left of B. Now let R denote the set of boundary primal vertices between A and B taken along with all the primal vertices in A. Now by Lemma 5.4, we can find a k-connection from B to k boundary primal vertices which are to the left of R. As we've done many a time, construct paths α_i and β_i in the normal fashion. We know need to show that these paths terminate. We summarize a way of doing it, but we don't go into enormous details because it's identical to the strategy used in [6].

First we observe that for each i, at least one of α_i and β_i must terminate, because if neither terminate we can define a trick function ψ which would have less power than ϕ. This forces all of the paths $\alpha_1, \beta_1, \alpha_2, \ldots, \alpha_{k-1}$ and β_{k-1} must terminate by planarity. Clearly they must terminate at vertices in A. We also know that at least one of the path α_k and β_k must terminate. A simple argument shows that all vertices in A must be "occupied" by vertices at the ends of paths $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k$. To show that both α_1 and β_1 terminate is identical to showing case 1 from the proof of Theorem 3.2.4 of [6]. The reader verifies that this impossible by planarity. The details are almost identical to the proof of Theorem 3.2.4 of [6]. □

6 New Results about Recoverability

Here we will begin work on answering some general questions about recoverability. As of yet, we do not have a general answer to recoverability for pseudocritical half planar graphs networks. Here we present several new results about pseudocritical networks. First we use Corollary 4.6 to show that a new class of graphs is recoverable. Then we give several examples of nontrivially nonrecoverable networks.

7 Partially-Supercritical Networks

Definition 7.1. Let $\Gamma = (G, \gamma)$ be an electrical network embedded into a partially-supercritical half planar medial graph with compact cells such that every edge is crossed by at least one compact geodesic. We call Γ **partially-supercritical**.

Lemma 7.2. Partially-supercritical networks have a least one boundary spike or boundary to boundary edge.

Proof. Simply take any reentrent geodesic, and perform the same trick as usual (taking a decreasing sequence of subsets of the medial graph which must terminate in a geodesic triangle). The details are left to the reader. □

A more useful way to phrase this is:

Proposition 7.3. Suppose M is a pseudocritical medial graph and g is a finite geodesic that intersects at least one other geodesic. Let $B(g)$ denote the set of cells in the compact region bounded by g. At least one of the cells in $B(g)$ is a boundary geodesic triangle.

Proof. The proof is identical to above, so it is omitted. □

Theorem 7.4. Partially-supercritical networks are recoverable from the minimal boundary data maps.

Proof. There are no new or profoundly insightful techniques used here. We basically will use Corollary 4.6 along with the tried and tested technique we used with finite graphs. Let e be an edge in G. We will show how to recover the conductivity along e. Let g be a finite geodesic which crosses e. By Proposition 7.3 we know that there is a geodesic triangle T somewhere in G. This geodesic triangle corresponds to either a boundary spike or a boundary to boundary edge. In either case, let the boundary to boundary edge or boundary spike be denoted by e'. Let g' be a finite geodesic which crosses this e'. Let p be the primal boundary vertex on e' which is in $B(g')$ (notice that this statement makes sense regardless of whether e' is a boundary spike or a boundary to boundary edge). By previous results there is a minimal function voltage-covoltage function defined on M which is supported only in $B(g)$ and which is equal to 1 at the primal vertex p. Furthermore, by Corollary 4.6 any minimal function with boundary values (both voltages and currents) supported in the boundary cells of $B(g')$, must be supported only on $B(g')$. The conductivity along e' can immediately be read off. Let Γ' be the electrical network corresponding to removing e' (by contracting if e' is a boundary spike and deleting if e' is a boundary to boundary edge). By previous results we can determine the minimal boundary data maps for Γ'. Removing the edge e' leaves all cells compact, leaves any finite geodesic finite, and doesn't add any loops or lenses. Finally, it removes exactly one cell from the region $B(g)$. By induction we must eventually exhaust all of $B(g)$, and thus recover the conductivity along the edge e. The details are nearly identical to the case of supercritical networks now that we have Corollary 4.6, so the details are left to the reader. □

Example 7.5. The graph in Figure 5 is obviously not recoverable, though it has been given a pseudocritical embedding.

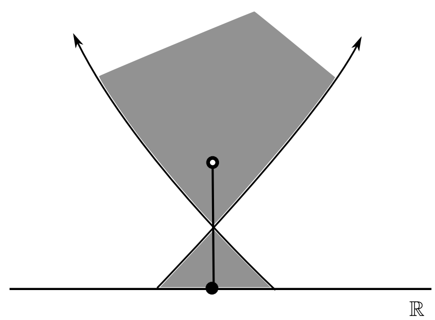

Figure 5: A trivially nonrecoverable pseudocritical half planar graph and its medial embedding into the half plane.

The issue is that there is ambiguity in which medial graph we embed the network into. In 5 the network could also be embedded into a medial graph with a self loop, which is "hidden" in the above medial graph by letting the geodesics go off to ∞.

Example 7.6. In Figure 6 is a less trivially example of a nonrecoverable graph and a medially embedding for it. We will call this network the $NR2$ network (**N**on**R**ecoverable 2).

We will leave the proof that this network is not recoverable for late.

We will need to work hard to show that this network is indeed not recoverable. To do this, we will need to introduce a little machinery.

7.1 Wye-Delta Transformations

In the finite case, we know that performing Wye-Delta transformations does not change the response matrix. We will want an analogue of this for infinite

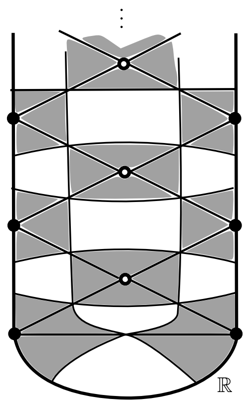

Figure 6: The $NR2$ network embedded in a medial graph. A less trivial example of a nonrecoverable pseudocritical half planar electrical network. Notice that two geodesics with domain $[0,1)$ intersect.

networks. It will actually turn out to be relatively straightforward to show the minimal Dirichlet-to-Neumann maps are unchanged by a finite number of Wye-Delta transformations. We will work to show that we can do the same thing for certain infinite sequences of Wye-Delta transformations, which will be necessary to show that certain graphs are not recoverable.

We first need a lemma from [6].

Theorem 7.7. Suppose $\Gamma = (G, \gamma)$ is an infinite electrical network and $\phi \in M(\Gamma)$ is a minimal function. Then we can write ϕ as a pointwise limit of γ-harmonic functions on an increasing chain of finite connected subnetworks of Γ which share the same boundary values on ϕ.

The proof can be found in [6], but is relatively straightforward.

Definition 7.8. Suppose that $\Gamma = (G, \gamma)$ and $\Gamma' = (G', \gamma')$ are electrical networks which have the same boundary vertices. We will say that Γ and Γ' are **electrically equivalent** if there is a map $\Phi : M(\Gamma) \to M(\Gamma')$ such that

(a) Φ is a linear bijection;

(b) $\Phi(\phi)|_{\partial G'} = \phi|_{\partial G}$;

(c) $\Lambda_{(M(\Gamma)}\phi = \Lambda_{M(\Gamma')}\Phi(\phi)$ (where $\Lambda_{M(\Gamma)}\phi$ is the ∂G-tuple of currents leaving the boundary from the minimal function Φ.

We will say that Γ and Γ' are **electrically isometric** if they are electrically equivalent and if in addition the map Φ is also an isometry (and hence an isometric isomorphism of Hilbert Spaces).

Theorem 7.9. Suppose that $\Gamma = (G, \gamma)$ and $\Gamma' = (G', \gamma')$ are electrical networks such that Γ' is obtained from Γ by a finite sequence of Wye-Delta transformations. Then Γ' and Γ are electrically isometric.

Proof. The proof goes by way of Theorem 7.7. Let G_0 be a connected finite subnetwork which contains all of the vertices and edges of G which are used in any of the Wye-Delta transformations used to pass from G to G'. Suppose that $\phi \in M(\Gamma)$. Note that the statement of the theorem is well known for finite networks. Complete G_0 to an exhausting increasing chain of finite subnetworks $G_0 \subseteq G_1 \subseteq \cdots$. Let G'_i denote the network G_i after applying the finite number of Wye-Delta transformations. Let ϕ_i be the γ-harmonic function on G_i with boundary values $\phi|_{\partial G_i}$ and let $\Phi_i : M(\Gamma_i) \to M(\Gamma'_i)$ be the map as in the finite case. By Theorem 7.7 some subsequence $\{\phi_{j_k}\}$ of $\{\phi_j\}$ converges to ϕ. Pick a subsequence of $\Phi_{j_k}(\phi_{j_k})$ which converges pointwise to a function ψ (take a diagonal subsequence; the details are left to the reader). By Fatou's lemma we know that ψ has finite power. Furthermore, an identical argument as in the proof of [6] of Theorem 7.7 shows that $\psi \in M(\Gamma')$. Thus we just define $\Phi(\phi)$ to be ψ. Since $\Phi_j(\phi_j)|_{\partial G'} = \phi_j|_{\partial G} = \phi|_{\partial G}$ we know that $\psi|_{\partial G'} = \phi|_{\partial G}$. Furthemore, we also know that $P(\Phi_j(\phi_j)) = P(\phi_j)$. Hence by Fatou's lemma we know that

$$P(\Phi(\phi)) = P(\psi) \leq P(\phi).$$

Now since all Wye-Delta transformations are invertable, we could just reverse this procedure, and perform all Wye-Delta transformations in the reverse order to go from G' to G. This would yield a map $\Psi : M(\Gamma') \to M(\Gamma)$ which preserves boundary values and satisfies (by what we've already shown) that $P(\Psi(\phi)) \leq P(\phi)$. Since $\phi|_{\partial G} = \psi|_{\partial G'}$, we would have that $\Psi(\Phi(\phi))$ and ϕ are both minimal functions with the same boundary values, and hence

$$\Psi(\Phi(\phi)) = \phi$$

. Hence

$$P(\Psi(\Phi(\phi))) = P(\phi)$$

and hence

$$P(\phi) \leq P(\Phi(\phi)).$$

Since we've already shown that $P(\Phi(\phi)) \leq P(\phi)$, we know that $P(\phi) = P(\Phi(\phi))$.

The final statement that the map Φ preserves currents on the boundary follows from observing that each Φ_i satisfies this property, and then just taking limits. □

Another way of stating this is as follows:

Corollary 7.10. If Γ and Γ' are electrical networks that are obtainable by a finite number of Wye-Delta transformations, then there are maps $\Phi : M(\Gamma) \to M(\Gamma')$ and $\Psi : M(\Gamma') \to M(\Gamma)$ such that

1. Φ and Ψ are isometries,

2. Φ and Ψ preserve boundary values

3. $\Phi \circ \Psi = \mathrm{id}|_{M(\Gamma')}$ and $\Psi \circ \Phi = \mathrm{id}|_{M(\Gamma)}$.

7.2 Passing an Edge Across a 4-star

We now describe an action which is the composition of two Wye-Delta transformations. We call it passing an edge across a 4-star. We describe it pictorially in Figure 7.

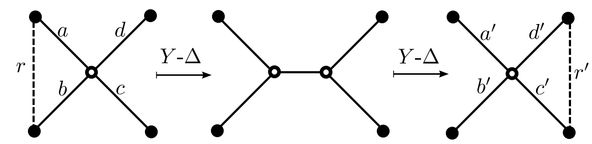

Figure 7: The operation of passing an edge across a 4-star by two consecutive Y-Δ equivalences.

The new conductivities after passing an edge across a 4-star can be explicitly computed to be

$$a' = \frac{ra + rb + ab}{b}, \tag{1}$$

$$b' = \frac{ra + rb + ab}{a},$$

$$c' = \left(\frac{ra + rb + ab}{r\sigma_0 + ab}\right) c,$$

$$d' = \left(\frac{ra + rb + ab}{r\sigma_0 + ab}\right) d,$$

and

$$r' = \frac{cdr}{r\sigma_0 + ab},$$

where

$$\sigma_0 = a + b + c + d.$$

Remark 7.11. Passing an edge across a 4-star is a composition of Wye-Delta transformations and hence induces an isometry between minimal functions on the network which preserves boundary values and boundary currents.

The operation of passing an edge across a 4-star has some pretty nice properties, which we go into now. A first remark is in order.

Definition 7.12. We will call the network in Figure 8 the **4+1-star** network. Notice that it is recoverable since the medial graph has no lenses.

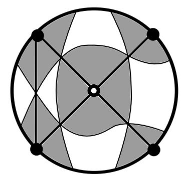

Figure 8: The augmented 4-star graph and its medial graph.

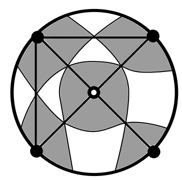

Figure 9: The augmented 4-star graph and its medial graph.

Definition 7.13. We will call the network in Figure 9 the **4+2-star** network. Notice that the $4+2$-star is recoverable since its medial graph has no lenses.

We now prove several easy theorems about passing an edge across a 4-star.

Proposition 7.14. The operation of passing an edge across a 4-star is **reversible**, i.e. the operation of passing an edge across a 4-star can be undone by passing an the resulting edge backwards across the same 4-star. This is shown in Figure 10.

Proof. This follows from the fact that Wye-Delta transformations are invertible.

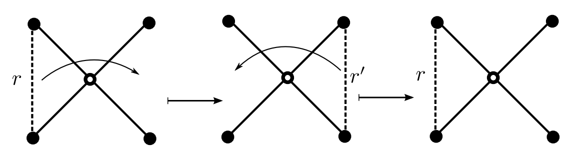

Figure 10: Passing an edge forward and then backward results in the original network.

□

Proposition 7.15. Passing an edge across a 4-star is **invertible**, i.e. passing conductivity r across a 4-star can be undone by passing a conductivity $-r$ in the same direction. This is shown in Figure 11.

Proof. This is just a direct computation from the equations in 1. We leave the computation to the reader.

□

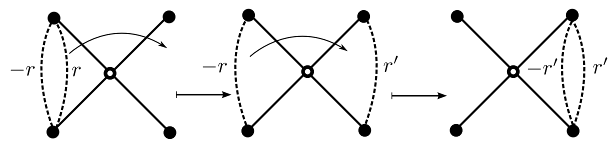

Figure 11: Passing r and then $-r$ results in no change to the network.

Proposition 7.16. Passing an edge across a 4-star does induces a natural map on the γ-harmonic functions on the network which does not influence the power on the network. This remains true even if the edge passed has negative conductivity (where the power function is just the formal sum $\sum \gamma_{vv'}(\phi(v) - \phi(v'))^2)$) so long as all the conductivities are defined in the resulting network.

Proof. This is left to the reader, but is a straightforward computation. □

Proposition 7.17. Passing an edge across a 4-star is **transversally commutative**, i.e. passing 2 edges across a 4-star doesn't depend on which they are passed if the two edges are on different but adjacent sides of the 4-star. As in Figure 12.

Proof. We use the recoverability of the 4+2-star network. Suppose we pass p and then q. This will produce the same graph as first passing q and then p. Since the graph produced will be a $4+2$ star, and the $4+2$-star graph is recoverable, and since the resulting graphs must be electrically equivalent, we know that the resulting networks must have the same electrical conductivities.

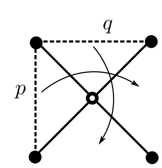

Figure 12: Passing edges in this manner is commutative.

□

We now will need to prove an extremely important but very tedious lemma. To do so, we will first introduce some terminology and notation.

7.3 A Technical Lemma

7.4 Nonrecoverability of the $NR2$ network

We will now show that the $NR2$ network is not recoverable. We begin with a basic fact about the $NR2$ network.

Lemma 7.18. Let Γ be the $NR2$ network. Then $H(\Gamma) = M(\Gamma)$, i.e. a finite power function is minimal iff it is γ-harmonic.

Proof. The only claim that requires proof is that any finite power γ-harmonic function is minimal. Let ϕ be a γ-harmonic function. Let W be the vector space of finite power functions which are supported only on the interior. We note that W is closed (by a theorem in [6]). Let W_f denote the vector space of finite power functions which are supported on $\operatorname{int} G$ which have finite support. Note that ϕ is orthogonal to every function W_f by assumption. By continuity of the inner product, we know that ϕ is orthogonal to everything in $\overline{W_f}$. Hence t show that ϕ is minimal (i.e. that ϕ is orthogonal to W) we will show that $\overline{W_f} = W$. Let $\psi \in W$. Give the interior vertices an arbitrary ordering $v_1, v_2, \ldots$. Let σ_i be the sum of the conductivities adjacent to v_i. Note that there are no interior vertices are adjacent, and hence by direct computation

$$\|\psi\|^2 = \sum_{i=1}^{\infty} \sigma_i \chi(v_i)^2.$$

Now define the function

$$\psi_N = \sum_{i=1}^{N} \chi_{v_i} \psi(v_i).$$

Note that $\psi_N \in W_f$ and furthermore

$$\|\psi_N - \chi\|^2 = \sum_{i=N+1}^{\infty} \sigma_i \chi(v_i)^2,$$

which goes to zero as $N \to \infty$, and hence $\psi_N \to \chi$ and hence $W = \overline{W_f}$, and the theorem statement follows. ☐

We note that the reader verifies that one can prove a slightly more general result that generalizes the above argument in the obvious way:

Corollary 7.19. Suppose that Γ is an (infinite) electrical network such that there are no interior-to-interior edges. Then $H(\Gamma) = M(\Gamma)$, i.e. all finite power functions are minimal.

Lemma 7.20. In the $NR2$ network, the γ-harmonic functions with finite support are dense in the set of minimal functions.

Proof. Let $\Gamma = (G, \gamma)$ be the $NR2$ network with some choice of conductivities. Let $M_f(\Gamma)$ denote the vector space of minimal functions on Γ with finite support. We wish to show that $M_f(\Gamma)$ is dense in $M(\Gamma)$. To do this, we will show that any element of $M(\Gamma)$ can be approximated arbitrarily well by an element of $M_f(\Gamma)$.

Let $\phi \in M(\Gamma)$. Let e denote the only boundary to boundary edge, and let S_n denote the n^{th} 4-star, as in Figure 13.

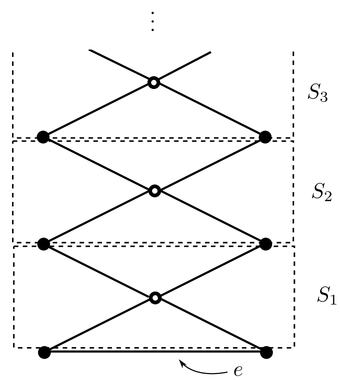

Figure 13: An enumeration of the 4-stars of Γ. The edge e is also shown.

Let $P_{S_i}(\phi)$ denote the power dissipated by ϕ on the 4-star S_i and let $P_e(\phi)$ denote the power dissipated by ϕ over the edge e. Now observe that

$$P(\phi) = P_e(\phi) + \sum_{i=1}^{\infty} P_{S_i}(\phi).$$

Now we will break each $P_{S_i}(\phi)$ further into two summands. Consider the labelling of the edges of the 4-star S_i shown in Figure 14.

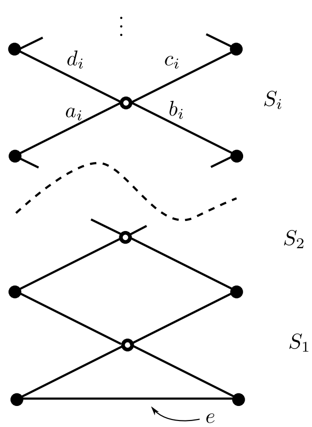

Figure 14: A labelling of the edges in the 4-star S_i.

Let $P^{\ell}_{S_i}(\phi)$ be the sum of the power dissipated on the edges a_i and b_i of the 4-star S_i, and let $P^{r}_{S_i}(\phi)$ be the sum of the power dissipated on the edges c_i and d_i of the 4-star S_i. Now note that $P_{S_i}(\phi) = P^{\ell}_{S_i}(\phi) + P^{r}_{S_i}(\phi)$ and hence

$$P(\phi) = P_e(\phi) + \sum_{i=1}^{\infty}(P^{\ell}_{S_i}(\phi) + P^{r}_{S_i}(\phi)). \tag{2}$$

Let $\epsilon > 0$. Since all of the summands in the above sum are positive and $P(\phi)$ is finite, pick N large enough that

$$\left| P(\phi) - \sum_{i=1}^{N} P_{S_i}(\phi) - P_e(\phi) \right| = P(\phi) - \sum_{i=1}^{N} P_{S_i}(\phi) - P_e(\phi) < \epsilon.$$

Using Equation (2) we get that this is equivalent to

$$0 \leq \sum_{i=N+1}^{\infty} P_{S_i}(\phi) < \epsilon \tag{3}$$

Let v_{N+1} denote the interior vertex of the 4-star S_{N+1}. By possibly adding the constant function to ϕ, we can assume that $\phi(v_{N+1}) = 0$. Now define the function $\phi_{N+1} : V \to \mathbb{R}$ by

$$\phi_{N+1}(v) = \begin{cases} \phi(v) & \text{if } v \in S_i \text{ and } i \leq n \\ 0 & \text{if } v = v_{N+1} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that ϕ_{N+1} has finite support and hence has finite power (notice however that ϕ_{N+1} may not be minimal!). Consider the function $\phi - \phi_{N+1}$. We simply compute to see that

$$(\phi - \phi_{N+1})(v) = \begin{cases} 0 & \text{if } v \in S_i \text{ and } i \leq n \\ 0 & \text{if } v = v_{N+1} \\ \phi(v) & \text{if } v \in S_i \text{ and } i > N+1. \end{cases}$$

We see immediately that

$$0 \leq P(\phi - \phi_{N+1}) = P^{r}_{S_{N+1}}(\phi) + \sum_{i=N+2}^{\infty} P_{S_i}(\phi) \leq \sum_{i=N+1}^{\infty} P_{S_i}(\phi).$$

Using Equation 3 we have that

$$P(\phi - \phi_{N+1}) < \epsilon. \tag{4}$$

Let $\Lambda : Z(\Gamma) \to M(\Gamma)$ be the map which projects finite power functions onto the unique minimal function with the same boundary values. As proven in [6], Λ is linear and the identity on $M(\Gamma)$, and furthermore $P(\Lambda(\psi)) \leq P(\psi)$. Applying these facts to Equation (4), we get that

$$\begin{aligned} P(\phi - \Lambda\phi_{N+1}) &= P(\Lambda\phi - \Lambda\phi_{N+1}) \\ &= P(\Lambda(\phi - \phi_{N+1})) \\ &\leq P(\phi - \phi_{N+1}) \\ &< \epsilon. \end{aligned}$$

Hence $P(\phi - \Lambda\phi_{N+1}) < \epsilon$. Notice that $\Lambda\phi_{N+1}$ is a minimal function with the same boundary values as ϕ_{N+1}. It's clear that if ψ is minimal function which is zero on the four boundary vertices of S_i, then ψ will be zero on the interior vertex of S_i. Since $\Lambda\phi_{N+1}$ is zero on the four boundary vertices of all but finitely many S_i, we know that $\Lambda\phi_{N+1}$ is zero on all but finitely many vertices, and hence in particular, $\Lambda\phi_{N+1} \in M_f(\Gamma)$. Since $P(\phi - \Lambda\phi_{N+1}) = \|\phi - \Lambda\phi_{N+1}\|^2 < \epsilon$, we know that $M_f(\Gamma)$ is dense in $M(\Gamma)$. □

Theorem 7.21. The $NR2$ network is not recoverable from the minimal boundary data map for any choice of conductivities. In particular, given an $NR2$ network Γ with any choice of conductivities, there is at least a one parameter family of $NR2$ networks which are electrically isometric.

Proof. Let $\Gamma = (G, \gamma)$ be the $NR2$ network with some choice of positive conductivities. Let γ_0 denote the conductivity along the bottom boundary to boundary edge (as shown in Figure 15).

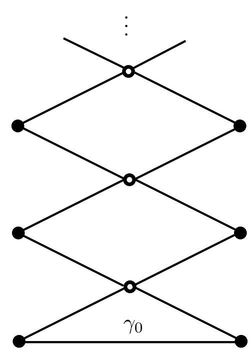

Figure 15: The conductivity γ_0 in the $NR2$ network.

Now let r be a real number $0 < r < \gamma_0$. Let e denote the boundary to boundary edge. We will construct an electrical network with the same graph G which has conductivity r on e which is electrically equivalent to Γ. To do this, split the edge e into two edge, e and e'. Put conductivity r on e and conductivity $\gamma_0 - r$ on e'. Notice that this network is electrically equivalent to Γ. See Figure 16.

Now pass the edge e' to infinity, i.e. pass it across the neighboring 4-star, and then pass it again across the next 4-star, and continue indefinitely, altering the conductivities in the way described in the section about passing an edge across a 4-star. This gives us another set of conductivities on the $NR2$ network. Let the network with the new conductivities be denoted by Γ'. We now need to show that these networks are electrically equivalent. We will show that there is a bijection from $M(\Gamma)$ to $M(\Gamma')$ which respects boundary voltages and boundary currents. Given a function $\phi \in M(\Gamma)$, after the i^{th} time we pass an edge across a 4-star, we get a new function, ϕ_i. Notice that at each vertex ϕ_i is eventually

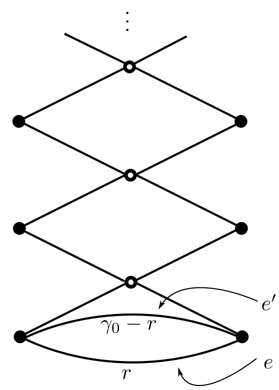

Figure 16: Splitting the edge e into two edges e and e'.

constant since Wye-Delta transformations only affect the voltages at the vertices involved. Let ψ be the pointwise limit of ϕ_i. A basic argument using Fatou's lemma shows that $P(\psi)$ is finite and furthermore $P(\psi) \leq P(\phi)$. Furthermore, it is easy to verify that ψ is γ-harmonic since each ϕ_i is γ-harmonic and the sequence ϕ_i is eventually constant at each vertex. By Lemma 7.18 we know that ψ is a minimal function. Define $\Phi(\phi)$ to be ϕ. By the uniqueness of minimal functions with given boundary values, we know that $\Phi(\phi)$ is well defined.

We will now show that Φ is an electrical isometry (a bijective isometry between $M(\Gamma)$ and $M(\Gamma')$ which respects boundary values and currents of minimal voltages). Clearly Φ is injective since minimal functions are uniquely determined by their boundary values, so we only need to show that it is surjective.

Suppose that $\psi \in M(\Gamma')$. First observe that Φ is continuous (as a map between Hilbert Spaces) since $P(\Phi(\phi)) \leq \phi$ (as observed by above). Let $M_f(\Gamma)$ denote the space of minimal functions on Γ with finite support and let $M_f(\Gamma')$ denote the space of minimal functions on Γ' with finite support. Note that if $\phi \in M_f(\Gamma)$, then once the edge e' is passed outside of the support of ϕ, the values and power of ϕ will no longer change. Since a finite sequence of Wye-Delta transformations is an electrical isometry, we know that $P(\Phi(\phi)) = \phi$, i.e. Φ is an isometry on $M_f(\Gamma)$. The reader verifies that boundary voltage function which is nonzero at only finitely many boundary vertices is valid for both networks, and hence Φ maps $M_f(\Gamma)$ bijectively (and isometrically) onto $M_f(\Gamma')$. By Lemma 7.20 we know that $M_f(\Gamma)$ is dense in $M(\Gamma)$ and that $M_f(\Gamma')$ is dense in $M(\Gamma)$. Since $M_f(\Gamma')$ is dense in $M(\Gamma')$, there is a sequence ψ_i of functions in $M_f(\Gamma')$ such that $\psi_i \to \psi$ in $Z(\Gamma)$ (i.e. that $P_{\Gamma'}(\psi - \psi_i) \to 0$. Since Φ maps $M_f(\Gamma)$ bijectively and isometrically onto $M_f(\Gamma')$, let ϕ_i denote $\Phi^{-1}(\psi_i)$. Since ψ_i converges to ψ, we know in particular that $\{\psi_i\}$ is a Cauchy sequence. Since Ψ is a bijective isometry from $M_f(\Gamma)$ onto $M_f(\Gamma')$ and both $M_f(\Gamma)$ and $M_f(\Gamma')$ are vector subspaces, we know that ϕ_i is a Cauchy sequence. Since Hilbert Spaces are complete (and $M(\Gamma)$ is a Hilbert space as it is a closed subspace of $Z(\Gamma)$, which is a Hilbert Space) we know that there is a $\phi \in Z(\Gamma)$ such that $\phi_i \to \phi$. Since $\phi_i \in M(\Gamma)$ and $M(\Gamma)$ is a closed subspace of $Z(\Gamma)$, we know that $\phi \in M(\Gamma)$. Since Φ is continuous (as $\Phi(\theta) \leq \theta$), we know that $\Phi(\phi_i) = \psi_i$ converges to $\Phi(\phi)$. Since $\psi_i \to \psi$ (and since Hilbert Spaces are Hausdorff topological spaces and hence limits are unique) we know that $\psi = \Phi(\phi)$, and hence Φ is bijective. Finally, by the continuity of the power functions, we know that

$$P(\psi_i) = P(\Phi(\phi_i)) = P(\phi_i) \to P(\phi).$$

Since $P(\psi_i) \to P(\psi)$, we know that $P(\psi) = P(\phi)$, and hence Φ is an isometry on all of $M(\Gamma)$. □

8 More Results About Electrical Equivalence

In this section we further develop our results about electrical equivalent of two networks. We will first prove a result about substituting electrically equivalent subnetworks of an electrical network.

8.1 Embedded Subnetworks

Definition 8.1. Let $\Gamma = (G, \gamma)$ and $\Sigma = (S, \sigma)$ be two electrical networks such that $S \subseteq G$ as unpartitioned graphs (meaning that the subset relation holds for the edge sets and vertex sets of S and G. We say that Σ is an embedded subnetwork of Γ (or just Σ is embedded in Γ) if the following conditions hold:

(i) $\operatorname{int} S \subseteq \operatorname{int} G$;

(ii) if $v \in \operatorname{int} S$ and $u \overset{G}{\sim} v$ then $u \in S$ and $u \overset{S}{\sim} v$;

(iii) if $e \in E(G) \cap E(S)$, then $\gamma(e) = \sigma(e)$.

Definition 8.2. Suppose that $\Sigma = (S, \sigma)$ is an embedded subnetwork of $\Gamma = (G, \gamma)$. Then we will define a map $R_0 : Z(\Gamma) \to Z(\Sigma)$ to be restriction of a function from in $Z(\Gamma)$ to the subnetwork Σ. Define R to be the restriction of R_0 to $M(\Gamma)$.

Proposition 8.3. Suppose that $\Sigma = (S, \sigma)$ is an embedded subnetwork of $\Gamma = (G, \gamma)$. Then R maps $M(\Gamma)$ into $M(\Sigma)$. Furthermore, R is a bounded linear operator of Hilbert Spaces.

Proof. Given a function $\phi \in M(\Gamma)$, it is clear that we should define

$$(R\phi)(v) = \phi(v)$$

for any $v \in S \subseteq G$. Clearly this is linear and satisfies $P(R\phi) \leq P(\phi)$ for any $\phi \in Z(\Gamma)$, and hence R is bounded.

We now just need to show that $R\phi \in M(\Sigma)$ whenever $\phi \in M(\Gamma)$. Suppose that $\phi \in M(\Gamma)$ but that $R\phi \notin M(\Sigma)$, then there exists a $\theta \in Z(\Sigma)$ such that

$\theta|_{\partial S} = R\phi|_{\partial S}$ but $P(\theta) < P(R\phi)$. Construct the function $\Theta : V(G) \to \mathbb{R}$ by defining

$$\Theta(v) = \begin{cases} \theta(v) & \text{if } v \in S \\ \phi(v) & \text{if } v \notin S. \end{cases}$$

We first will show that $\Theta|_{\partial G} = \phi|_{\partial G}$. Suppose that $v \in \partial G$. Then in particular, by property (i) of the definition of an embedded subgraph, we know that $v \notin \operatorname{int} S$. Hence if $v \in S$, we know that $\Theta(v) = \theta(v) = \phi(v)$ and if $v \notin S$ we know that $\Theta(v) = \phi(v)$ by construction. Hence $\Theta|_{\partial G} = \phi|_{\partial G}$.

We now will show that $P(\Theta) < P(\phi)$. This is simply a computation. We compute

$$\begin{aligned} P(\Theta) &= \sum_{v,v' \in V(G)} \gamma_{v,v'}(\Theta(u) - \Theta(v))^2 \\ &= \sum_{u,v \notin S} \gamma_{v,v'}(\Theta(u) - \Theta(v))^2 \\ &\quad + 2\left(\sum_{u \notin S; v \in S} \gamma_{v,v'}(\Theta(u) - \Theta(v))^2\right) \\ &\quad + \sum_{u,v \in S} \gamma_{v,v'}(\Theta(u) - \Theta(v))^2 \end{aligned}$$

.

Now observe that by condition (ii) of an embedded subnetwork, if $u \in S$ and $v \notin S$ and $u \overset{G}{\sim}$ then $u \in \partial S$. Hence $\Theta(u) = \theta(u) = \phi(u)$. Hence we can rewrite the above equation to see that

$$\begin{aligned} P(\Theta) &= \sum_{v,v' \in V(G)} \gamma_{v,v'}(\phi(u) - \phi(v))^2 \\ &= \sum_{u,v \notin S} \gamma_{v,v'}(\phi(u) - \phi(v))^2 \\ &\quad + 2\left(\sum_{u \notin S; v \in S} \gamma_{v,v'}(\phi(u) - \phi(v))^2\right) \\ &\quad + \sum_{u,v \in S} \gamma_{v,v'}(\theta(u) - \theta(v))^2. \end{aligned}$$

Noticing that the last summand is $P(\theta)$ we get

$$\begin{aligned}
P(\Theta) &= \sum_{u,v\notin S} \gamma_{v,v'}(\phi(u)-\phi(v))^2 + 2\left(\sum_{u\notin S; v\in S} \gamma_{v,v'}(\phi(u)-\phi(v))^2\right) + P(\theta) \\
&< \sum_{u,v\notin S} \gamma_{v,v'}(\phi(u)-\phi(v))^2 + 2\left(\sum_{u\notin S; v\in S} \gamma_{v,v'}(\phi(u)-\phi(v))^2\right) + P(R\phi) \\
&= P(\phi).
\end{aligned}$$

Hence $P(\Theta) < P(\phi)$, which contradicts the fact that ϕ was minimal. Hence R maps $M(\Gamma)$ into $M(\Sigma)$, as we wanted. □

To be absolutely pedantic, we now remark that we will always view R as having codomain $M(\Sigma)$ instead of $Z(\Sigma)$.

8.2 Splicing Subnetworks

We now will describe a fairly intuitive operation of taking an embedded electrical subnetwork $\Sigma \subseteq \Gamma$ and an electrically isometric subnetwork $\Sigma' \cong \Sigma$ and replacing Σ in Γ with Σ', to produce a new network Γ' which is electrically isometric to Γ.

Definition 8.4. Suppose $\Sigma = (S, \sigma), \Sigma' = (S', \sigma')$, and $\Gamma = (G, \gamma)$ are all electrical networks and Σ is an electrical subnetwork of Γ and Σ and Σ' are electrically isometric. Define the network the electrical network (T, τ) with vertex set $(G \setminus S) \cup S'$ and adjacency defined as follows:

(i) if $u, v \in G \setminus S$ then $u \overset{T}{\sim} v$ iff $u \overset{G}{\sim} v$;

(ii) if $u \in G \setminus S$ and $v \in S'$, then $u \overset{T}{\sim} v$ iff $v \in \partial S'$ (note that $\partial S = \partial S' \subseteq G$ by construction) and $u \overset{G}{\sim} v$;

(iii) if $u, v \in S'$ then $u \overset{T}{\sim} v$ iff $u \overset{S'}{\sim} v$.

We define the conductivity function $\tau : E(T) \to \mathbb{R}^+$ by

$$\tau(uv) = \begin{cases} \gamma(uv) & \text{if } u, v \in G \setminus S \\ \gamma(uv) & \text{if } u \in G \setminus S, v \in \partial S' (= \partial S \subseteq G) \\ \sigma'(uv) & \text{if } u, v \in S' \end{cases}.$$

Finally we define $\partial T = \partial G$ and $\operatorname{int} T = T \setminus \partial T$. Note that since $\partial S' = \partial S$ and $\partial G \subseteq (G \setminus S) \cup \partial S$ since Σ is embedded in Γ, we know that ∂G is indeed a subset of T, so the above definition makes sense.

We call the network (T, τ) **the network Γ with S and S' spliced** and we write $\Gamma(\Sigma : \Sigma')$ for (T, τ).

We now state and prove the culminating theorem of this section, namely that spliced networks are electrically isometric.

Theorem 8.5. Suppose $\Gamma = (G, \gamma), \Sigma = (S, \sigma), \Sigma' = (S', \sigma')$ are electrical networks such that Σ is embedded in Γ and such that Σ is electrically isometric to Σ'. Then Γ is electrically isometric to $\Gamma(\Sigma : \Sigma')$. Furthermore, S' is an embedded subnetwork of $\Gamma(\Sigma : \Sigma')$ and we have the following commutative diagram of Hilbert Space maps:

$$\begin{array}{ccc} M(\Gamma) & \xrightarrow{\cong} & M(\Gamma(\Sigma : \Sigma')) \\ \downarrow R & & \downarrow R \\ M(\Sigma) & \xrightarrow{\cong} & M(\Sigma') \end{array}$$

Proof. Let $\Phi : M(\Sigma) \to M(\Sigma')$ be an electrical isometry and let $\Theta : M(\Sigma') \to M(\Sigma)$ be its inverse (and hence also an electrical isometry). Let $\Gamma' = (G', \gamma')$ denote the network $\Gamma(\Sigma : \Sigma')$. We will define a map Ξ between $Z(\Gamma)$ and $Z(\Gamma(\Sigma : \Sigma'))$ in the following way. Suppose $\phi \in Z(\Gamma)$. Define $\Xi\phi$ by

$$(\Xi\phi)(v) = \begin{cases} \phi(v) & \text{if } v \in G \setminus S \\ (\Phi\Lambda_\Sigma(\phi|_S))(v) & \text{if } v \in S \end{cases}.$$

The reader checks that indeed $(\Xi\phi)|_{\partial G'} = \phi|_{\partial G}$. Furthermore, one checks that $P(\Xi\phi) \leq P(\phi)$.

We need only show that Ξ maps $M(\Gamma)$ into $M(\Gamma(\Sigma : \Sigma'))$. We note that Σ' is an embedded subnetwork of Γ', and that the spliced network $\Gamma'(\Sigma' : \Sigma)$ is actually equal to Γ. Let $H : Z(\Gamma') \to Z(\Gamma)$ be the splice map from Γ' to Γ defined analogously to Ξ. Once again we have that H satisfies $P(H\psi) \leq P(\psi)$. Now let Λ_Γ denote the canonical projection of $Z(\Gamma)$ onto $M(\Gamma)$ which preserves boundary values and similarly let $\Lambda_{\Gamma'}$ denote the projection of $Z(\Gamma')$ onto $M(\Gamma')$. Suppose that $\phi \in M(\Gamma)$. We wish to show that $\Xi\phi \in M(\Gamma')$. Observe that we have the following string of inequalities:

$$P(\phi) \geq P(\Xi\phi) \geq P(\Lambda_{\Gamma'}\Xi\phi) \geq P(H\Lambda_{\Gamma'}\Xi\phi) \geq P(\Lambda_\Gamma H\Lambda_{\Gamma'}\Xi\phi). \tag{5}$$

But now we observe that all of the maps $\Lambda_\Gamma, H, \Lambda_{\Gamma'}$ and Ξ preserve boundary values. Hence we know that $(\Lambda_\Gamma H\Lambda_{\Gamma'}\Xi)\phi$ is a minimal function with the same boundary values as ϕ and hence

$$\Lambda_\Gamma H\Lambda_{\Gamma'}\Xi\phi = \phi.$$

In particular $P(\Lambda_\Gamma H\Lambda_{\Gamma'}\Xi\phi) = P(\phi)$. This implies that we have equality all throughout Equation 5 and hence $P(\Xi\phi) = P(\Lambda_{\Gamma'}\Xi\phi)$. But this implies that $\Xi\phi$ has the same power as a minimal function with the same boundary values, which implies that $\Xi\phi$ is minimal. Hence Ξ maps $M(\Gamma)$ into $M(\Gamma')$. Furthermore, one shows in the exact same manner that the map H defined above maps $M(\Gamma')$ into $M(\Gamma)$, is an isometry on $M(\Gamma')$ and satisfies $\Xi \circ H = \mathrm{id}_{M(\Gamma')}$ and $H \circ \Xi = \mathrm{id}_{M(\Gamma)}$.

The fact that the diagram in the theorem statement commutes is obvious and follows from our construction. □

We now state a generalization of the maximum principle.

Proposition 8.6 (Strong Maximum Principle)**.** If $\phi \in M(\Gamma)$ then $\phi|_H$ satisfies the maximum principle for any embedded subnetwork $H \subseteq \Gamma$.

Proof. The function $\phi|_H$ is a minimal function on H and hence satisfies the maximum principle. □

8.3 Electrical Scheaves

9 More Examples of Nonrecoverable Networks

Definition 9.1. Define the **infinite 4-star strip network** or **4SS** network to be the network in Figure 17.

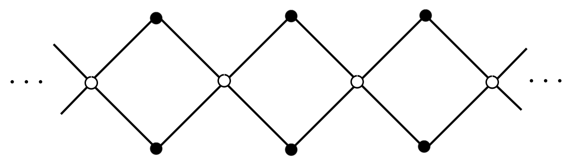

Figure 17: The infinite 4-star strip network, or $4SS$.

Theorem 9.2. The $4SS$ network is in general not recoverable. In particular, there exists a one parameter family of conductivities on the network which are electrically isometric. Fix a $K > 0$ and let t and s be any positive numbers such that $t + s = K$. Consider a modified 4SS network shown in Figure 18.

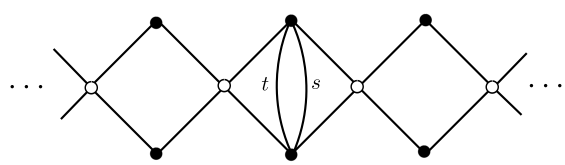

Figure 18: The modified $4SS$ network.

Proof. For any choice of s and t such that $t + s = K$ and $t, s > 0$, the above networks are electrically isometric. By passing the edge with conductivity t to infinity to the left and by passing the edge with conductivity s to infinity to the right, we get $4SS$ networks which are electrically isometric to our original modified $4SS$ network (this is actually nontrivial, but using our analysis of the $NR2$ network and the fact that the there are two copies of the $NR2$ embedded in the modified $4SS$ network, the reader easily checks the details). For different choices of s and t we get different conductivities on the network. Hence we have a one parameter family of $4SS$ networks which are electrically isometric.

□

Definition 9.3. We will define the **ultralattice 1** or **UL1 network** to be the network in Figure 19.

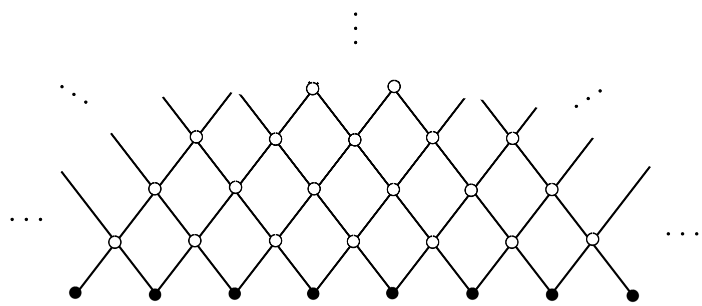

Figure 19: The first ultralattice network, UL1.

Theorem 9.4. The UL1 network is not recoverable.

Proof. The UL1 network has the 4SS network as an embedded subnetwork. □

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